Postdoc Fellowships for non-EU researchers

Final Report

Name	Hasan A. Fallahgoul
Selection	2013
Host institution	ULB
Supervisor	Prof. Davy Paindaveine
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Title	Inference for Tempered Stable Distribution

1. Objectives of the Fellowship (1/2 page)

Stable and tempered stable distributions have been used for solving many practical problems (see, e.g., Kim et al. (2012), Rachev et al. (2011), and Fallahgoul et al. (2012)). The application to finance dates backs to the Mandelbrot (1961, 1963). Prior to his body of work, the usual assumption was that the distribution of price changes in a speculative series is approximately Gaussian. In a financial time series context, this means that the distribution of the innovations (white noise) in a model - such as an autoregressive moving average of return series - is normally distributed. Mandelbrot showed that financial assets rather exhibit leptokurtosis and skewness, which is not compatible with Gaussianity. Mandelbrot used the stable distribution, which he referred to as the *stable Paretian distribution*, to exclude the special normal distribution case and to formulate a more suitable model for describing financial asset returns. Subsequently, the (heavy tailed) *stable Paretian* and *tempered stable distributions* are now the most popular alternatives to the normal distribution in finance.

Recently, stable distributions have increasingly been replaced by tempered stable distributions in financial applications. The major reasons for this are the following: (1) tempered stable distributions have finite and exponential moments, (2) they have heavier tails than the normal distribution, and thinner than stable distributions, and (3) goodness-of-fit tests show that tempered stable distributions are more suitable for fitting data. However, most studies have focused on the univariate case and very few works considered the multivariate case. Rosinski (2007) considered a general class of multivariate tempered stable distributions, while Baeumer and Kovacs (2011) proposed a method for approximating multivariate tempered stable distributions. The objectives of the initial proposal then could be summarized as follows:

1. Defining multivariate (elliptical) tempered stable distributions and studying its properties.

2. Proposing estimation methods for the parameters of univariate and multivariate elliptical tempered stable distributions.

3. Applying the resulting methodology to multivariate heavy tailed data, in particular in finance.

2. Methodology in a nutshell (1/2 page)

Stable and tempered stable distributions belong to the class of heavy tailed distributions. They are very useful for modeling tail risk. One of the problems with these distributions is that their probability density function (PDF) and cumulative distribution function (CDF) do not have a closed-form formula. This makes of course estimation more challenging. However, their characteristic function possesses a closed-form expression. In this project, we overcome the issue of their estimation and extend such distributions to the elliptical case.

In order to tackle the issue of estimation, we use the method of simulated quantiles.[1] In this approach, we construct functions of quantiles, which are informative about the parameters of interest. Next, the estimated values for related parameters are obtained by minimizing the Mahalanobis distance between theoretical and simulated functions of quantiles. Simulated functions of quantiles are obtained through characteristic functions.

All classes of stable and tempered stable distributions are included in the class of Levy processes. They can be considered as time-changed processes. In fact, by changing the time of the arithmetic Brownian motion with some positive non-decreasing process (subordinator), one can obtain different classes of Levy processes. In this project, by changing the time of the multivariate Brownian motion with classical and rapidly decreasing tempered stable subordinators, we introduce the elliptical tempered stable distributions. In this approach, the characteristic function is still available in closed form. Therefore, we can rely on simulated-based methods to estimate the parameters of elliptical tempered stable distributions.

3. Results (6-8 pages)

Results for the first objective

Regarding the first objective, we proposed a suitable form of multivariate tempered stable distributions, which we referred to as the *elliptical tempered stable (ETS) distributions*. We introduced some analytic approximations for the PDF for a subclass of the ETS distributions, namely the *tempered infinite divisible (TID) distributions*. A definition for ETS copula was also discussed. Moreover, first, we introduced a definition for the ETS distribution based on a unique spectral measure along with some of its theoretical properties. Second, a framework for linking fractional calculus and TID was provided. More specifically, some fractional partial differential equations (PDE) were introduced whose fundamental solution gives the entire family for the TID's pdf. Third, the analytic approximations for the PDF of the TID and ETS

[1] Detailed information about the method of simulated quartile can be found in Dominicy and Veredas (2013).

distributions were derived. Finally, a definition for the ETS copula was considered, as well as providing an algorithm for its numerical evaluation.

In this first objective, we linked probability theory and fractional calculus. More specifically, we introduced some classes of fractional partial differential equations whose fundamental solution gives the pdf of TID and ETS distributions. By using some analytic approaches such as the Homotopy Perturbation Method (HPM), Adomian Decomposition Method (ADM) and Variational Iteration Method (VIM), an analytic approximation for the pdf of the tempered infinite divisible and elliptical tempered distributions was obtained. However, these approximations have a singularity in the origin, which causes some difficulties when applying them in practice. A possible way to overcome this problem is to use mixture distributions.

We provided a new strategy for obtaining an analytic approximation of the PDF for the multivariate stable and geo-stable distributions. This strategy is an extension of Fallahgoul et al. (2012) for the univariate case. Specifically, we employed three analytic approximation methods — homotopy perturbation method, Adomian decomposition method, and variational iteration method — to compute the fundamental solutions of a PDE of fractional order (e.g., Momani and Odibat 2007; Elsaid 2010). These three methods offer efficient approaches for solving linear and nonlinear PDEs (e.g., Sweilam and Khader 2009), integral equations (e.g., Bougoffa et al. 2012), and integro-differential equations (e.g., El-Kalla 2012) that have been applied to a wide class of problems in physics, biology, and chemical reaction. The key in our presentation is that multivariate stable and geo-stable distributions are linked to partial differential equations of fractional order. After introducing new partial differential equations of fractional order that are related to multivariate stable and geo-stable distributions, we then derived analytical-numeric formulas for nearly all the PDFs of the multivariate stable and geo-stable distributions.

Because elliptical tempered stable distributions do not have a closed-form formula for their probability density function and cumulative density function, a numerical procedure is needed for calculating them. We provided more details about generating elliptical tempered stable random vectors and the calculation of both the probability density function and cumulative density function of elliptical tempered stable distributions. Based on this numerical process, we simulated an elliptical tempered stable copula and a tail dependency coefficient.

Results for the second objective

We introduced a simple, fast, and accurate method to estimate the parameters of tempered stable probability distributions. Estimation is based on the *Method of Simulated Quantiles (MSQ)* and consists of matching empirical and theoretical functions of quantiles that are informative about the parameters of interest. In a Monte Carlo study, we showed that MSQ is significantly faster than Maximum Likelihood Estimation (MLE) and that the corresponding estimates are almost as precise as MLE. A Value at Risk (VaR) study using 13 years of daily returns from 21 world-wide market indexes showed that MSQ estimates provide as good risk assessments as with MLE.

Since the MSQ method is based on quantiles, it does not make use of the frequency domain. And since it is based on simulations, we do not need closed-form expressions of any function that represents the probability law of the process. In a nutshell, MSQ is based on a vector of functions of quantiles that are informative about the parameters of interest. These functions can be either computed from data (the sample functions) or from the distribution (the theoretical functions). The estimated parameters are those that minimize a quadratic distance between both. Since the theoretical functions of quantiles may not have a closed-form expression, we rely on simulations.

We focused on three tempered stable distributions: the classical one, the generalized classical one, and the normal one. Henceforth we refer to them as classical tempered stable, generalized tempered stable and normal tempered stable, respectively. For each distribution, we proposed functions of quantiles that are informative about the unknown parameters. While the general asymptotic theory of MSQ applies to any distribution, we studied the finite sample performances of the resulting estimators through a Monte Carlo study. The estimates are unbiased and their root-mean square errors (RMSEs) are low. We also showed that the MSQ methods brings computational advantages with respect to the MLE: estimation based on quantiles takes roughly half the time used by estimation based on likelihoods.

The finite-sample performances of the estimators can be summarized in two points. First, for the three classes of the tempered stable distributions and all configurations considered, median estimates are close to the true values and the RMSEs are small for all parameters (implying that variances are small, too). Second, while the advantage of sequences of quantiles with respect to functions of quantiles is the automatization, Monte Carlo results showed that the advantage of functions of quantiles over sequences of quantiles is twofold: (i) functions of quantiles estimates are more accurate, both in terms of bias and RMSEs. (ii) The computational cost with functions of quantiles is lower than with sequences of quantiles, which is not surprising as the number of functions in functions of quantiles is typically much smaller than the grid size q used for sequences of quantiles.

We illustrated the estimation method with 13 years of daily stock log returns for 21 major world- wide equity market indexes, covering America, Europe, and Asia and Oceania. Our first and most important finding is that the relative efficiencies of the MSQ estimates (relative to MLE) are very close to one. Put it differently, MQS estimators are almost as efficient as MLE. Additionally, our VaR analysis and backtest showed that the estimated VaRs are reasonable. In fact, they are as good as with MLE.

Three main conclusions could be drawn from the empirical study. First, parameters are in general estimated accurately. For the classical tempered stable, the parameters that control the rate of decay on the positive and negative tails are in most cases statistically different at 5% significance level. The same happens for the normal tempered stable distribution. The skewness parameter is most of the times negative and statistically different from zero for more than 50% of the market indexes. Second, for all estimated parameters, the relative efficiencies are close to one, which shows that MSQ (and more particularly the approach based on functions of quantiles) is not only faster than MLE, but also provides estimators that almost reach the Cramer-Rao lower bound. Third, the null hypothesis of correct distributional specification cannot be rejected at 5% significance level for all markets and for the classical tempered stable and normal tempered stable distributions. This is why we do not estimate the generalized tempered stable. Moreover, the p-values for the KS test of the normal tempered stable are generally larger than for the classical tempered stable, suggesting that the latter distribution provides a better fit.

We performed a battery of backtesting VaR tests. The first set of tests are the three Christoffersen (1998) likelihood ratio (CLR) tests (CLRuc for unconditional coverage, CLRind for independence, and CLRcc for coverage and independence). The second set are two Berkowitz (2001) likelihood ratio (BLR) tests of independence (denoted BLRind) and of accuracy of forecasts of the tail distribution (denoted BLRtail). As opposed to CLR tests, BLR's do not count the number of violations but instead use the inverse of the standard normal cumulative distribution function and estimated cumulative distribution function.

We considered the CTS and NTS distributions, with parameters estimated with MLE and MSQ. Since all the results for the 21 market indexes were too numerous to report, we showed the risk performance for the main market indexes of three continents (FTSE, Nikkei and S&P500).

The CTS and NTS distributions provided accurate VaR estimations, even in years with a great deal of turmoil like 2008, though we observed eight rejections (with significance level 5%) of the null hypotheses (five using MLE and three using MSQ estimates). Among these rejections, seven are for BLRind, which seems to be the most restrictive test statistic, and one using MLE and the CLRuc test. Overall, we conclude that VaR assessments using ML and MSQ estimates are equally good. And given that MSQ are almost as efficient as MLE and much faster to compute, we have built a case for the use of MSQ for the estimation of tempered stable distributions.

Results for the third objective

Some parts of the second objective and third objective are not complete yet. In particular, we still need to develop a methodology for the estimation of elliptical tempered stable distributions, and to apply it model to financial data. This is considered for further investigation in the future collaboration between centers.

4. Perspectives for future collaboration between units (1 page)

As just mentioned, we still need to introduce a new method for the estimation of elliptical tempered stable distributions. This is under further investigation. This part of project as well as the investigation of the performances of elliptical tempered stable distributions in modeling the risk of financial data are considered for future collaboration between the units.

The main contributions of this part are three-fold. First and foremost, the method does not suffer from the curse of dimensionality, and hence can be applied to vast dimensions. It is also fast and accurate. Second, it can be used for any tempering function, providing a flexibility not available until now. Third, it does not require the use of derivatives, nor an educated guess of the initial values.

In a nutshell, the method is again based on quantiles and works as follows. First, the parameters are split into three distinct subsets. The first subset gathers the parameters that regulate the tail behavior. They are estimated by the Method of Simulated Quantiles (MSQ). Due to the small dimension of this subset (usually, two or three), optimization can be performed with a grid search, which allows to overcome to the problems related with gradient-based methods. The second subset gathers the parameters that form the dispersion matrix. By means of results in McCulloch (1986), projections, and the properties of elliptical distributions, the estimators of the dispersions and co-dispersions have a closed form. The third subset is the vector of locations, which are estimated in a straightforward way (and also in closed form).

5. Valorisation/Diffusion (including Publications, Conferences, Seminars, Missions abroad...

Some results of this project are published and the others have been grouped in two working papers :

1. H. A. Fallahgoul, Y. S. Kim, F. J. Fabozzi, Elliptical Tempered Stable Distribution, *Quantitative Finance*, (2016), forthcoming.

2. H. A. Fallahgoul, D. Veredas, F. J. Fabozzi, Quantile-base Inference for Tempered Stable Distribution, (2015), *SFI@EPFL Working Paper*.

3. H. A. Fallahgoul, D. Veredas, Inference for Vast Dimensional Elliptical Tempered Stable Distribution, *work in progress*, (2016).

6. Skills/Added value transferred to home institution abroad (1/2 page)

Simulation-based methods show good performances for point estimation. Such approached are more valuable when the traditional methods such as maximum likelihood method are not applicable. Since the stable and tempered stable distribution do not have closed-form formula for the probability density function and cumulative distribution function, the estimation of them are not straightforward, which makes simulation-based methods of particular interest.

The host institute has high experience and extensive knowledge in the field of simulation-based methods, which allowed to give Hasan Fallahgoul training in the field. These methods can be used for tackling the estimation problem of the stable and tempered stable distributions. By transferring these skills to his home institute, Hasan Fallahgoul might foster further collaborations between the units in the field.